

# Multipartite entanglement and high-precision metrology

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We present several entanglement criteria in terms of the quantum Fisher information that help to relate various forms of multipartite entanglement to the sensitivity of phase estimation. We show that genuine multipartite entanglement is necessary to reach the maximum sensitivity in some very general metrological tasks using a two-arm linear interferometer. We also show that it is needed to reach the maximum average sensitivity in a certain combination of such metrological tasks.

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## I. INTRODUCTION

There has been a rapid development in the technology of quantum experiments with photons [1–6], trapped ions [7,8], and cold atoms [9]. In many of the experiments the goal is to create a state with genuine multipartite entanglement [1–8]. From the operational point of view, the meaning of such an entanglement is clear [7,10]. An  $N$ -qubit quantum state with genuine  $N$ -partite entanglement cannot be prepared by mixing  $N$ -qubit pure states, in which some groups of particles have not interacted. Thus, the experiment presents something qualitatively new compared to an  $(N - 1)$ -qubit experiment. There is an extensive literature on the detection of such entanglement (e.g., see Ref. [11] for a review).

One of the important applications of entangled multipartite quantum states is sub-shot-noise metrology [12]. In metrology, as can be seen in Fig. 1, one of the basic tasks is phase estimation connected to the unitary dynamics of a linear interferometer

$$\rho_{\text{output}} = e^{-i\theta J_{\vec{n}}} \rho e^{+i\theta J_{\vec{n}}}, \quad (1)$$

where  $\rho$  is the input state of the interferometer, while  $\rho_{\text{output}}$  is the output state, and  $J_{\vec{n}}$  is a component of the collective angular momentum in the direction  $\vec{n}$ . The important question is how well we can estimate the small angle  $\theta$  measuring  $\rho_{\text{output}}$ . For such an interferometer the phase estimation sensitivity, assuming *any* type of measurement, is limited by the quantum Cramér-Rao bound as [13,14]

$$\Delta\theta \geq \frac{1}{\sqrt{F_Q[\rho, J_{\vec{n}}]}}, \quad (2)$$

where  $F_Q$  is the quantum Fisher information. The relationship between phase estimation sensitivity and entanglement in linear interferometers has already been examined [15], and an entanglement condition has been formulated with the sensitivity of the phase estimation, that is, with the quantum Fisher information. It has been found that some entangled states provide a better sensitivity in phase estimation than separable states. It has also been proven that not all entangled states are useful for phase estimation, at least in a linear interferometer [16]. Moreover, in another context, it has been noted that multipartite entanglement, not only simple

nonseparability, is needed for extreme spin squeezing [17]. While this finding is not directly related to the theory of the quantum Fisher information, it does show that multipartite entanglement is needed for a large sensitivity in certain concrete metrological tasks.

A question arises: Would it be possible to relate genuine multipartite entanglement or any other type of multipartite entanglement to the quantum Fisher information? Apart from the point of view of metrology, this is also interesting from the point of view of entanglement criteria. Bipartite entanglement criteria generalized for the multipartite case typically detect any, that is, not necessarily genuine, multipartite entanglement. In fact, so far conditions for genuine multipartite entanglement were mostly linear in operator expectation values (e.g., entanglement witnesses [18–21] or Bell inequalities [22–26]). There are also criteria quadratic in operator expectation values [27–29], a strong criterion based on the elements of the density matrix [30,31] and some spin squeezing inequalities [32–34]. For us, a starting point can be that existing entanglement conditions based on the Wigner-Yanase skew information [35] can also detect genuine multipartite entanglement and many properties of the skew information and the quantum Fisher information are similar.

In this paper, we examine what advantage states with various forms of multipartite entanglement offer over separable states in metrology. We show that in order to have the maximal sensitivity in certain metrological tasks,  $\rho$  must be genuinely multipartite entangled. It can also happen that an entangled state does not provide a sensitivity for phase estimation larger than the sensitivity achievable by separable states for any  $J_{\vec{n}}$ ; however, the average sensitivity of phase estimation is still larger than for separable states. Thus, when asking about the advantage of entangled states over separable ones in phase estimation, it is not sufficient to consider the sensitivity in a single metrological task.

Now we are in a position to formulate our first main results; the proofs are given later.

*Observation 1.* For  $N$ -qubit separable states, the values of  $F_Q[\rho, J_l]$  for  $l = x, y, z$  are bounded as

$$\sum_{l=x,y,z} F_Q[\rho, J_l] \leq 2N. \quad (3)$$

Here  $J_l = \frac{1}{2} \sum_{k=1}^N \sigma_l^{(k)}$ , where  $\sigma_l^{(k)}$  are the Pauli spin matrices for qubit ( $k$ ). Later we also show that Eq. (3) is a condition

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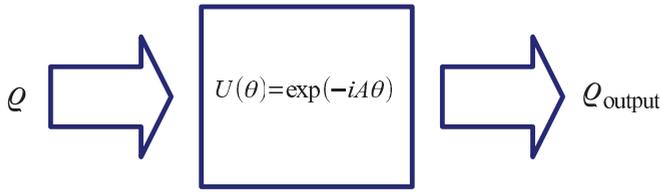


FIG. 1. (Color online) The basic problem of linear interferometry. The parameter  $\theta$  must be estimated by measuring  $\rho_{\text{output}}$ .

for the average sensitivity of the interferometer. All states violating Eq. (3) are entangled. Note that, according to Ref. [15], for separable states we have

$$F_Q[\rho, J_l] \leq N. \quad (4)$$

*Observation 2.* For quantum states, the quantum Fisher information is bounded by above as

$$\sum_{l=x,y,z} F_Q[\rho, J_l] \leq N(N+2). \quad (5)$$

Greenberger-Horne-Zeilinger states (GHZ states, [36]) and  $N$ -qubit symmetric Dicke states with  $\frac{N}{2}$  excitations saturate Eq. (5). Note that the above symmetric Dicke state has been investigated recently due to its interesting entanglement properties [4,6,32]. It has also been noted that the above Dicke state gives an almost maximal phase measurement sensitivity in two orthogonal directions [16]. In general, pure symmetric states for which  $\langle J_l \rangle = 0$  for  $l = x, y, z$  saturate Eq. (5).

Next we consider  $k$ -producible states [35,37]. A pure state is  $k$  producible if it is a tensor product of at most  $k$ -qubit states [37]. A mixed state is  $k$  producible if it is a mixture of pure  $k$ -producible states.

*Observation 3.* For  $N$ -qubit  $k$ -producible states, the quantum Fisher information is bounded from above by

$$F_Q[\rho, J_l] \leq nk^2 + (N - nk)^2, \quad (6)$$

where  $n$  is the integer part of  $\frac{N}{k}$ . A condition similar to Eq. (6) has appeared in Ref. [35] for the Wigner-Yanase skew information.

*Observation 4.* For  $N$ -qubit  $k$ -producible states, the sum of three Fisher information terms is bounded from above by [38]

$$\begin{aligned} & \sum_{l=x,y,z} F_Q[\rho, J_l] \\ & \leq \begin{cases} nk(k+2) + (N-nk)(N-nk+2) & \text{if } N-nk \neq 1, \\ nk(k+2) + 2 & \text{if } N-nk = 1, \end{cases} \end{aligned} \quad (7)$$

where  $n$  is the integer part of  $\frac{N}{k}$ . Any state that violates this bound is not  $k$  producible and contains  $(k+1)$ -particle entanglement.

Next we consider criteria that show that the quantum state is not biseparable. A pure state is biseparable if it can be written as a tensor product of two multipartite states [10]. A mixed state is biseparable if it can be written as a mixture of biseparable pure states. The bounds for biseparable states for the left-hand-side of Eqs. (6) and (7) can be obtained from Observations 3 and 4 after taking  $n = 1$  and maximizing the bounds in those

Observations over  $k = \lceil \frac{N}{2} \rceil, \lceil \frac{N}{2} \rceil + 1, \dots, N-1$ , where  $\lceil \frac{N}{2} \rceil$  is the smallest integer not smaller than  $\frac{N}{2}$ . Hence, we obtain

$$F_Q[\rho, J_l] \leq (N-1)^2 + 1, \quad (8a)$$

$$\sum_{l=x,y,z} F_Q[\rho, J_l] \leq N^2 + 1. \quad (8b)$$

Any state that violates Eqs. (8a) or (8b) is genuine multipartite entangled.

The inequalities presented in Observations 1–3 correspond to planes in the  $(F_Q[\rho, J_x], F_Q[\rho, J_y], F_Q[\rho, J_z])$  space as can be seen in Fig. 1 for  $N = 6$  particles. These observations show that for fully separable states only a very small fraction of the  $(F_Q[\rho, J_x], F_Q[\rho, J_y], F_Q[\rho, J_z])$  space is allowed. This is also true for states with several forms of multipartite entanglement, for example,  $k$ -producible states with  $k \ll N$ . To reach the maximal phase sensitivity, genuine multipartite entanglement is needed.

The paper is organized as follows. In Sec. II, we prove Observations 1 and 2. In Sec. III, we prove Observations 3 and 4. In Sec. IV, we examine the characteristics of the states corresponding to interesting points in the  $(F_Q[\rho, J_x], F_Q[\rho, J_y], F_Q[\rho, J_z])$  space and determine which regions correspond to quantum states of different degrees of entanglement. In Sec. V, we discuss some similarities to entanglement detection with uncertainty relations. In Appendix A, we present a unified framework to derive entanglement conditions independent from the coordinate system chosen. In Appendix B, we give some details of our calculations.

## II. PROOF OF OBSERVATIONS 1 AND 2

First, let us review some of the central notions concerning metrology and the quantum Fisher information. For calculating many quantities, it is sufficient to know the following two relations [13–15,39] for the quantum Fisher information.

(1) For a pure state  $\rho$ , we have  $F[\rho, J_l] = 4(\Delta J_l)_\rho^2$ .

(2)  $F[\rho, J_l]$  is convex in the state; that is,  $F[p_1\rho_1 + p_2\rho_2, J_l] \leq p_1 F[\rho_1, J_l] + p_2 F[\rho_2, J_l]$ .

From these two statements, it also follows that  $F[\rho, J_l] \leq 4(\Delta J_l)_\rho^2$ .

There is also an explicit formula for computing the quantum Fisher information for a given state  $\rho$  and a collective observable  $J_{\vec{n}}$  for any  $\vec{n}$  as [16]

$$F_Q[\rho, J_{\vec{n}}] = \vec{n}^T \Gamma_C \vec{n}. \quad (9)$$

Thus, the  $\Gamma_C$  matrix carries all the information needed to compute  $F_Q[\rho, J_{\vec{n}}]$  for any direction  $\vec{n}$ . It is defined as [16]

$$[\Gamma_C]_{ij} = 2 \sum_{l,m} (\lambda_l + \lambda_m) \left( \frac{\lambda_l - \lambda_m}{\lambda_l + \lambda_m} \right)^2 \langle l | J_i | m \rangle \langle m | J_j | l \rangle, \quad (10)$$

where the sum is over the terms for which  $\lambda_l + \lambda_m \neq 0$ , and the density matrix has the decomposition

$$\rho = \sum_k \lambda_k |k\rangle \langle k|. \quad (11)$$

Note that for pure states  $[\Gamma_C]_{ij} = \langle J_i J_j + J_j J_i \rangle / 2 - \langle J_i \rangle \langle J_j \rangle$  [16]. Later, we present entanglement conditions with  $\Gamma_C$ , besides entanglement conditions with  $F[\varrho, J_l]$ .

*Proof of Observation 1.* First we show that Observation 1 is true for pure states. We use here the theory of entanglement detection based on uncertainty relations [40]. According to this theory, for every  $N$ -qubit pure product state of the form

$$|\Psi_p\rangle = \otimes_{n=1}^N |\Psi_n\rangle, \quad (12)$$

the variance of the collective observable  $J_l$  is the sum of the variances of the single-qubit observables  $j_l^{(n)} = \frac{1}{2}\sigma_l^{(n)}$  for the single-qubit states  $|\Psi_n\rangle$ . Thus, we have for the sum of the variances of the three angular momentum components [41]

$$\begin{aligned} \sum_{l=x,y,z} (\Delta J_l)_{|\Psi_p\rangle}^2 &= \frac{1}{4} \sum_{l=x,y,z} \sum_{n=1}^N (\Delta \sigma_l)_{|\Psi_n\rangle}^2 \\ &= \frac{1}{4} \sum_{n=1}^N (3 - \langle \sigma_x^{(n)} \rangle^2 - \langle \sigma_y^{(n)} \rangle^2 - \langle \sigma_z^{(n)} \rangle^2) = \frac{N}{2}. \end{aligned}$$

For the mixture of product states, that is, for mixed separable states, Eq. (3) follows from the convexity of the Fisher information. ■

Next we show that Eq. (3) can be interpreted as a condition on the average sensitivity of the interferometer. First, note that Eq. (3) can be reformulated with the eigenvalues of  $\Gamma_C$  as

$$\text{Tr}(\Gamma_C) \leq 2N. \quad (13)$$

Then, using Eq. (9), we obtain

$$\text{avg}_{\vec{n}}(F_Q[\varrho, J_{\vec{n}}]) = \text{avg}_{\vec{n}}\{\text{Tr}[\Gamma_C(\vec{n}\vec{n}^T)]\} = \text{Tr}(\Gamma_C \frac{\mathbb{1}}{3}), \quad (14)$$

where averaging is over all three-dimensional unit vectors. Thus, Eq. (3) can be rewritten as a condition for the average sensitivity of the interferometer as

$$\text{avg}_{\vec{n}}(F_Q[\varrho, J_{\vec{n}}]) \leq \frac{2}{3}N. \quad (15)$$

Let us calculate now the maximum of the left-hand side of Eq. (3).

*Proof of Observation 2.* We have to use that the quantum Fisher is never larger than the corresponding variance,

$$\sum_{l=x,y,z} F(\varrho, J_l) \leq 4 \sum_{l=x,y,z} (\Delta J_l)^2, \quad (16)$$

and that the sum of the variances are bounded from above

$$4 \sum_{l=x,y,z} (\Delta J_l)^2 \leq 4 \sum_{l=x,y,z} \langle J_l^2 \rangle \leq N(N+2). \quad (17)$$

For pure states, Eq. (16) is saturated. The second inequality of Eq. (17) appears as a fundamental equation in the theory of angular momentum. For symmetric states with  $\langle J_l \rangle = 0$  for  $l = x, y, z$ , both inequalities of Eq. (17) are saturated. Hence, GHZ states and Dicke states with  $\frac{N}{2}$  excitations saturate Eq. (5). ■

### III. BOUNDS FOR MULTIPARTITE ENTANGLEMENT

In this section we present the proof of Observations 3 and 4 and also compute some bounds for other types of multipartite entanglement. For that, we use ideas similar to the ones in the proof of Observation 1.

*Proof of Observation 3.* Let us consider pure states that are the tensor product of at most  $k$ -qubit entangled states,

$$|\Psi_{k\text{-producible}}\rangle = |\Psi_1^{(N_1)}\rangle \otimes |\Psi_2^{(N_2)}\rangle \otimes |\Psi_3^{(N_3)}\rangle \otimes |\Psi_4^{(N_4)}\rangle \otimes \dots, \quad (18)$$

where  $N_m \leq k$  is the number of qubits for the  $m$ th term in the product. Hence, based on using  $(\Delta J_l)_{|\Psi_m^{(N_m)}\rangle}^2 \leq \frac{N_m^2}{4}$  for the  $N_m$ -qubit units, we obtain

$$(\Delta J_l)_{|\Psi_{k\text{-producible}}\rangle}^2 = \sum_m (\Delta J_l)_{|\Psi_m^{(N_m)}\rangle}^2 \leq \sum_m \frac{N_m^2}{4}.$$

For the case when  $k$  is a divisor of  $N$ , the largest variance can be obtained for a state for which all  $N_m = k$ . Hence, for the state Eq. (18) we obtain

$$(\Delta J_l)^2 \leq \frac{N}{k} \times \frac{k^2}{4}. \quad (19)$$

If  $k$  is not a divisor of  $N$  then at least one of the states in the tensor product of Eq. (18) will have fewer than  $k$  qubits. The maximum for the sum of the variances is obtained if all but a single state has  $k$  qubits. Considering this, we obtain Eq. (6). The strong dependence of the bounds on  $k$  in Eq. (6) indicates that for high-precision metrology states containing many-partite entanglement are needed. ■

*Proof of Observation 4.* Let us consider pure states that are the tensor product of at most  $k$ -qubit entangled states of the form Eq. (18) Hence, based on using Eq. (5) for the  $k$ -qubit units, we obtain

$$\begin{aligned} \sum_{l=x,y,z} (\Delta J_l)_{|\Psi_{k\text{-producible}}\rangle}^2 &= \sum_m \sum_{l=x,y,z} (\Delta J_l)_{|\Psi_m^{(N_m)}\rangle}^2 \leq \sum_m \frac{N_m(N_m+2)}{4}. \end{aligned} \quad (20)$$

For the case when  $k$  is a divisor of  $N$ , the largest variance can be obtained for a state for which all  $N_m = k$ . Hence, for the state Eq. (18) we obtain

$$\sum_{l=x,y,z} (\Delta J_l)^2 \leq \frac{N}{k} \frac{k(k+2)}{4}. \quad (21)$$

If  $k$  is not a divisor of  $N$ , then at least one of the states in the tensor product of Eq. (18) will have fewer than  $k$  qubits. The maximum for the sum of the variances is obtained if all but a single state has  $k$  qubits. Considering this, we obtain Eq. (7). We have to use that for pure states of  $N \geq 2$  qubits, we have  $\sum_k (\Delta J_l)^2 \leq \frac{N(N+2)}{4}$ , while for  $N = 1$  we have a better bound  $\sum_k (\Delta J_l)^2 \leq \frac{1}{2}$ . ■

*Bound for states with a given number unentangled particles.* Next, we obtain bound for systems that contain a given number of unentangled particles. A pure state is told to contain  $M$  unentangled particles if it can be written as [37,42]

$$\otimes_{k=1}^M |\Psi_k\rangle \otimes |\Psi_{M+1,\dots,N}\rangle. \quad (22)$$

We say that a mixed state contains at least  $M$  unentangled particles if it can be prepared by mixing pure states with  $M$  or more unentangled particles.

Many interesting quantum states are highly entangled, but still contain only two-particle entanglement. Nevertheless, it is still important to know how large fraction of the particles remain unentangled since the number of unentangled particles is directly connected to metrological usefulness of the state.

*Observation 5.* For states with at least  $M$  unentangled particles, the quantum Fisher information is bounded from above by

$$\sum_{l=x,y,z} F_Q[\varrho, J_l] \leq M + (N - M)(N - M + 2). \quad (23)$$

*Proof.* For a pure state of the form Eq. (22), we have

$$\sum_{l=x,y,z} (\Delta J_l)^2 \leq \frac{M}{4} + \frac{(N - M)(N - M + 2)}{4}. \quad (24)$$

Any state that violates Eq. (23) has fewer than  $M$  unentangled particles. The validity of Eq. (23) for mixed states is due to the convexity of the quantum Fisher information. ■

So far, we presented entanglement conditions in terms of  $F_Q[\varrho, J_l]$  for  $l = x, y, z$ . A more general approach is constructing entanglement conditions with the  $\Gamma_C$  matrix defined in Eq. (10). In Appendix A, we present unified framework for determining entanglement conditions for  $\Gamma_C$ .

#### IV. INTERESTING POINTS IN THE $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])$ SPACE

In this section, we discuss which part of the  $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])$  space contains points corresponding to states with different degrees of entanglement. This is important since, apart from finding inequalities for states of various types of entanglement, we have to show that there are states that fulfill these inequalities.

For that, let us see first the interesting points of the  $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])$  space and the corresponding quantum states, shown in Fig. 2.

- (i) A completely mixed state,

$$\varrho_C = \frac{\mathbb{1}}{2^N}, \quad (25)$$

corresponds to the point  $C(0, 0, 0)$  in the  $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])$  space.

- (ii) Product states of the form

$$|\Psi\rangle_{S_l} = |+\frac{1}{2}\rangle_l^{\otimes N/2} \otimes |-\frac{1}{2}\rangle_l^{\otimes N/2} \quad (26)$$

for  $l = x, y, z$  correspond to the points  $S_x(0, N, N)$ ,  $S_y(N, 0, N)$ , and  $S_z(0, N, N)$ , respectively.

- (iii) An  $N$ -qubit symmetric Dicke state with  $\frac{N}{2}$  excitations in the  $z$  basis is defined as

$$|\mathcal{D}_N^{(N/2)}\rangle = \binom{N}{N/2}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \{ |0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \}, \quad (27)$$

where  $\sum_k \mathcal{P}_k$  denotes summation over all possible permutations. Such a state corresponds to the point  $D_z(\frac{N(N+2)}{2}, \frac{N(N+2)}{2}, 0)$ . Dicke states in the  $x$  and  $y$  bases correspond to the points  $D_x(0, \frac{N(N+2)}{2}, \frac{N(N+2)}{2})$  and  $D_y(\frac{N(N+2)}{2}, 0, \frac{N(N+2)}{2})$ , respectively.

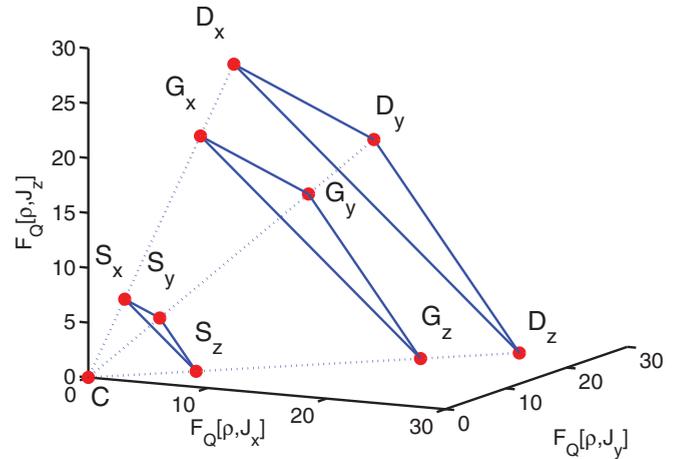


FIG. 2. (Color online) Interesting points in the  $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])$  space for  $N = 6$  particles. Points corresponding to separable states satisfy Eq. (3) and are not above the  $S_x$ - $S_y$ - $S_z$  plane. Points corresponding to biseparable states satisfy Eq. (8b) and are not above the  $G_x$ - $G_y$ - $G_z$  plane. All states corresponding to points above the  $G_x$ - $G_y$ - $G_z$  plane are genuine multipartite entangled. For the coordinates of the  $S_l$ ,  $G_l$ ,  $D_l$ , and  $C$  points, see Sec. IV.

- (iv) An  $N$ -qubit GHZ state in the  $z$  basis is defined as

$$|\Psi\rangle_{GHZ_z} = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}). \quad (28)$$

It corresponds to the point  $(N, N, N^2)$ . GHZ states in the  $x$  and  $y$  bases correspond to points  $(N^2, N, N)$  and  $(N, N^2, N)$ , respectively.

- (v) Finally, the tensor product of a single-qubit state and a Dicke state of the form

$$|\Psi\rangle_{G_z} = |1\rangle \otimes |\mathcal{D}_{N-1}^{N/2-1}\rangle \quad (29)$$

corresponds to the point  $G_z(\frac{N^2}{2} + \frac{1}{2}, \frac{N^2}{2} + \frac{1}{2}, 0)$  [43]. States corresponding to the points  $G_x$  and  $G_y$  can be obtained from  $|\Psi\rangle_{G_z}$  by basis transformations. After considering individual points, we now show that there are two-dimensional objects in the  $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])$  space such that for all of their points there is a corresponding separable or entangled quantum state.

- (vi) For all points in the  $S_x, S_y, S_z$  polytope, there is a corresponding pure product state for even  $N$ . Given  $F[\varrho, J_l]$  for  $l = x, y, z$ , such a state is defined as

$$\varrho = \left[ \frac{\mathbb{1}}{2} + \frac{1}{2} \sum_{l=x,y,z} c_l \sigma_l \right]^{\otimes N/2} \otimes \left[ \frac{\mathbb{1}}{2} - \frac{1}{2} \sum_{l=x,y,z} c_l \sigma_l \right]^{\otimes N/2}, \quad (30)$$

where  $c_l^2 = 1 - \frac{F_Q[\varrho, J_l]}{N}$ , where  $\sum_l c_l^2 = 1$ .

- (vii) For all points in the  $D_x, D_y, D_z$  polytope, there is a corresponding quantum state if  $N$  is divisible by 4. To see this, let us consider the following quantum states for even  $N$ :

$$|\Psi_{\text{even}}\rangle = \sum_{n=0,2,4,\dots,N/2-2} c_n \frac{1}{\sqrt{2}} (|\mathcal{D}_N^{(n)}\rangle + |\mathcal{D}_N^{(N-n)}\rangle) + c_{N/2} |\mathcal{D}_N^{(N/2)}\rangle, \quad (31)$$

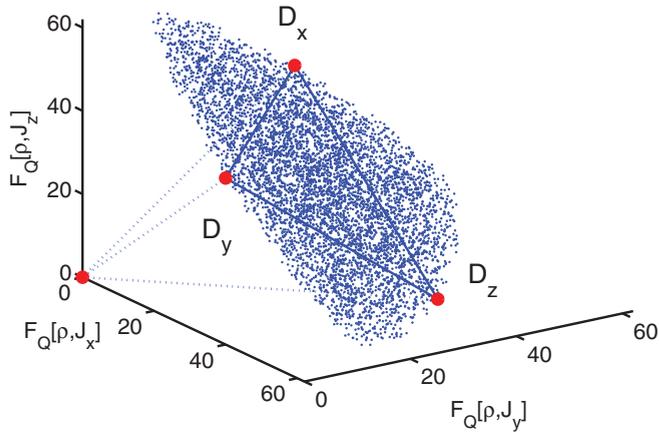


FIG. 3. (Color online) Randomly chosen points in the  $(F_Q[\rho, J_x], F_Q[\rho, J_y], F_Q[\rho, J_z])$  space corresponding to states of the form Eq. (32) for  $N = 8$ . All the points are in the plane of  $D_x$ ,  $D_y$ , and  $D_z$ .

where  $c_n$  are complex coefficients. States Eq. (31) are special cases of symmetric states with an even parity [44]. For  $|\Psi_{\text{even}}\rangle$ , we have  $\langle J_l \rangle = 0$  for  $l = x, y, z$ . Finally,  $\langle J_l J_m + J_m J_l \rangle = 0$  if  $l \neq m$ ; thus, for  $|\Psi_{\text{even}}\rangle$  the matrix  $\Gamma_C$  is diagonal. Let us now assume that  $N$  is a multiple of 4 and consider the states of the form

$$|\Psi(\alpha_x, \alpha_y, \alpha_z)\rangle = \alpha_x |\mathcal{D}_N^{(N/2)}\rangle_x + \alpha_y |\mathcal{D}_N^{(N/2)}\rangle_y + \alpha_z |\mathcal{D}_N^{(N/2)}\rangle_z, \quad (32)$$

where  $\alpha_l$  are complex coefficients. (Note that  $|\mathcal{D}_N^{(N/2)}\rangle_l$  are not pairwise orthogonal.) Simple analytical calculations show that such states are a subset of the states Eq. (31) [45]. The states (32) fill the polytope  $D_x$ ,  $D_y$ , and  $D_z$ , which is demonstrated for  $N = 8$  in Fig. 3 [46] (see also Appendix B). Thus, there is a quantum state corresponding to all points of this polytope.

Next we examine, how to obtain states corresponding to three-dimensional polytopes. For that we use that when mixing two states, the points corresponding to the mixed state are on a curve in the  $(F_Q[\rho, J_x], F_Q[\rho, J_y], F_Q[\rho, J_z])$  space. In the general case, this curve is not a straight line. For the case of mixing a pure state with the completely mixed state, the curve is a straight line. Such a state is defined as

$$\rho^{(\text{mixed})}(p) = p\rho + (1-p)\frac{\mathbb{1}}{2^N}. \quad (33)$$

Using Eq. (10), after simple calculations we have

$$\Gamma_C^{(\text{mixed})}(p) = \frac{p^2}{p + (1-p)2^{-(N-1)}} \Gamma_C^{(\rho)}. \quad (34)$$

Hence, we can state the following.

*Observation 6.* If  $N$  is even, then there is a separable state for each point in the  $S_x, S_y, S_z, C$  polytope.

*Proof.* This is because there is a pure product state corresponding to any point in the  $S_x, S_y, S_z$  polytope. When mixing any of these states with the completely mixed state, we obtain states that correspond to points on the line connecting the pure state to point  $C$ . ■

*Observation 7.* If  $N$  is divisible by 4, then for all the points of the  $D_x, D_y, D_z, G_x, G_y, G_z$  polytope, there is a quantum state with genuine multipartite entanglement.

*Proof.* There is a quantum state for all points in the  $D_x, D_y, D_z$  polytope. Mixing them with the completely mixed state, states corresponding to all points of the  $C, D_x, D_y, D_z$  polytope can be obtained. Based on Observation 2, states corresponding to the points in the  $D_x, D_y, D_z, G_x, G_y, G_z$  polytope are genuine multipartite entangled. ■

Finally, note that all the quantum states we presented in this section have a diagonal  $\Gamma_C$  matrix. Thus, our statements remain true even if the three coordinate axes in Fig. 2 correspond to the three eigenvalues of  $\Gamma_C$ .

## V. DISCUSSION

The criterion in Eq. (3) contains several quantum Fisher information terms. It can happen that a state does not violate the criterion Eq. (4), but it violates the criterion Eq. (3). In this case, for a single metrological task of the type we considered in this paper its entanglement does not make it possible to outperform the metrology with separable states. However, if the state is used for several metrological tasks, then it makes it possible to achieve such an *average* sensitivity that would be not possible for separable states.

A related example is the proposal of using multipartite singlets for differential magnetometry [47]. Singlets are useful for differential magnetometry because they are insensitive to homogeneous fields, that is,  $F_Q[\rho, J_l] = 0$  for  $l = x, y, z$ , which is the same as for the completely mixed state. However, when considering operators other than  $J_{\vec{n}}$ , singlets turn out to be very sensitive, which is not the case for the completely mixed state. Thus, singlets can provide an advantage over separable states if the combination of two metrological tasks are considered.

It is instructive to compare the necessary condition for separability Eq. (3) to the condition presented in Refs. [41,48],

$$\sum_{l=x,y,z} (\Delta J_l)^2 \leq \frac{N}{2}. \quad (35)$$

Clearly, if a pure state is detected by Eq. (35), it is not detected by Eq. (3), and vice versa. In fact, Eqs. (35) and (3) together detect all entangled pure multiqubit states except for the ones for which

$$\sum_{l=x,y,z} (\Delta J_l)^2 = \frac{N}{2}. \quad (36)$$

Of course, the two conditions also detect some mixed entangled states in the vicinity of the pure entangled states.

It is an interesting question whether multipartite states having a positive partial transpose for all bipartitions can violate any of the above entanglement criteria with the quantum Fisher information. Violating Eq. (3) would certainly mean that such bound entangled states are useful for certain metrological applications. To find such states, if they exist, might be difficult as typically bound entangled states are strongly mixed and the quantum Fisher information is convex.

Concerning multipartite entanglement, Observation 3 shows that for a single metrological task, genuine multipartite entanglement is needed to reach the maximum sensitivity.

Observation 4 demonstrates that even for the maximum average sensitivity for the metrological tasks considered can be reached only by states possessing genuine multipartite entanglement.

Finally, the definition of quantum Fisher information used in Ref. [15], while widely considered “the” quantum Fisher information, is not the only possible definition [49]. The Wigner-Yanase skew information is another possibility [50–52]. This quantity equals the variance for pure states, and it is also convex in the state. This has already been used to define entanglement criteria with the skew information [35,53]. Thus, all previous statements can easily be transformed into criteria with the skew information.

## VI. CONCLUSIONS

In summary, we showed that genuine multipartite entanglement, or in general, multipartite entanglement more demanding than simple inseparability, is needed to achieve a maximal accuracy using multipartite quantum states for metrology. We also considered several relations with the quantum Fisher information and determined the corresponding bounds for various forms of entanglement.

*Note added in proof.* Independently from our work, another paper on the relationship between multipartite entanglement and Fisher information has been prepared [54].

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## APPENDIX A: ENTANGLEMENT CONDITIONS FOR THE $\Gamma_C$ MATRIX

In this Appendix, we present a unified framework to derive entanglement conditions for the  $\Gamma_C$  matrix. For that aim, we use ideas from the derivation of the covariance matrix criterion [55,56] and the entanglement criteria for Gaussian multimode states [57,58]. We recall that a separable state is a mixture of pure product states [59],

$$\rho_{\text{sep}} = \sum_k p_k \rho_{\text{pure product},k}. \quad (\text{A1})$$

Due to the convexity of the quantum Fisher information [15], we have

$$F[\rho_{\text{sep}}, J_{\vec{n}}] \leq \sum_k p_k F[\rho_{\text{pure product},k}, J_{\vec{n}}]. \quad (\text{A2})$$

Thus, for every separable state there must be a set of  $p_k$  and  $\rho_{\text{pure product},k}$  fulfilling Eq. (A2). Hence, we can say the

following. For every separable state, there is a set of  $p_k$  and  $\rho_{\text{pure product},k}$  such that

$$\Gamma_C^{(\text{sep})} \leq \sum_k p_k \Gamma_C^{(\text{pure product},k)}. \quad (\text{A3})$$

Any state for which there are not such a set of probabilities and pure product density matrices is entangled [60].

It is known that for  $N$ -qubit pure product states we have the following two constraints for the variances of the three angular momentum components,

$$\sum_{l=x,y,z} (\Delta J_l)^2 = \frac{N}{2}, \quad (\text{A4a})$$

$$(\Delta J_m)^2 \leq \frac{N}{4}, \quad (\text{A4b})$$

which has been used to derive entanglement criteria with the three variances [41,42,48,61]. Equation (A4a) also appeared in the proof of Observation 1. Based on Eq. (A4), the conditions for the eigenvalues of  $\Gamma_C^{(\text{pure product})}$  are clearly

$$\begin{aligned} \sum_{l=x,y,z} \Lambda_l^{(\text{pure product})} &= 2N, \\ 0 &\leq \Lambda_m^{(\text{pure product})} \leq N \end{aligned} \quad (\text{A5})$$

for  $m = x, y, z$ . Using now our knowledge about  $\Gamma_C^{(\text{pure product},k)}$ , the condition Eq. (A3) leads to the following equations for the eigenvalues of  $\Gamma_C^{(\text{sep})}$ :

$$\sum_{l=x,y,z} \Lambda_l^{(\text{sep})} \leq 2N, \quad (\text{A6a})$$

$$0 \leq \Lambda_m^{(\text{sep})} \leq N, \quad (\text{A6b})$$

for  $m = x, y, z$ . Equation (A6) can be reformulated with  $\Gamma_C$  as

$$\text{Tr}(\Gamma_C^{(\text{sep})}) \leq 2N, \quad (\text{A7a})$$

$$\Lambda_{\max}(\Gamma_C^{(\text{sep})}) \leq N, \quad (\text{A7b})$$

where  $\Lambda_{\max}(A)$  is the largest eigenvalue of  $A$ . Equation (A7b) has appeared in Ref. [16].

Hence, quantum states fulfilling Eq. (A3) must fulfill Eq. (A7). In Observation 1 and also for the criterion Eq. (4), the most entangled states are detected if  $F[\rho_{\text{sep}}, J_l]$  correspond to the three eigenvalues of  $\Gamma_C$ . For this case, Eq. (A7a) is equivalent to Observation 1 and Eq. (A7b) is equivalent to Eq. (4).

In a similar manner, conditions for multipartite entanglement can also be obtained. Thus, analogously to Observation 3 and Observation 4, for  $N$ -qubit  $k$ -producible states, we obtain

$$\begin{aligned} \text{Tr}(\Gamma_C^{(\text{sep})}) &\leq \begin{cases} nk(k+2) + (N-nk)(N-nk+2) & \text{if } N-nk \neq 1, \\ nk(k+2) + 2 & \text{if } N-nk = 1, \end{cases} \\ &\quad (\text{A8a}) \end{aligned}$$

$$\Lambda_{\max}(\Gamma_C^{(\text{sep})}) \leq nk^2 + (N-nk)^2, \quad (\text{A8b})$$

where  $n$  is the largest integer such that  $nk \leq N$ . We can obtain the bounds for biseparability setting  $n = 1$  and

$k = N - 1$ . Any state that violates one of the criteria for  $n = 1$  and  $k = N - 1$  is genuine multipartite entangled. The inequalities (A8a) and (A8b) are essentially the criteria of Observations 3 and 4 rewritten in a coordinate system independent way.

### APPENDIX B: $\Gamma_C$ MATRIX FOR THE STATE EQ. (32)

In this Appendix, we compute the  $\Gamma_C$  matrix for the superposition of three Dicke states given in Eq. (32). We show that for any point in the  $D_x, D_y, D_z$  triangle in Fig. 3 there is a corresponding state of this type.

First we need to know that

$${}_k \langle \mathcal{D}_N^{(N/2)} | J_l^2 | \mathcal{D}_N^{(N/2)} \rangle_m = \begin{cases} \frac{N(N+2)}{8} & \text{if } k = m \neq l, \\ Q & \text{if } k \neq m \text{ and } m \neq l \text{ and } k \neq l, \\ 0 & \text{otherwise} \end{cases} \quad (\text{B1})$$

for  $k, l, m \in \{x, y, z\}$ . In the second line on the right-hand side of Eq. (B1),  $Q = {}_x \langle \mathcal{D}_N^{(N/2)} | J_y^2 | \mathcal{D}_N^{(N/2)} \rangle_z$ . Since the state vector of  $|\mathcal{D}_N^{(N/2)}\rangle_x$  and  $|\mathcal{D}_N^{(N/2)}\rangle_z$  all have real elements, and  $J_y^2$  also have only real elements for even  $N$ ,  $Q$  is also real. Its precise value is not important for proving the main statement of this section. The last line on the right-hand side of Eq. (B1) is due to the fact that  $J_l |\mathcal{D}_N^{(N/2)}\rangle_l = 0$ .

Hence, the  $\Gamma_C$  matrix for state Eq. (32) is a diagonal matrix, with

$$\Gamma_{C,xx} = (|\alpha_y|^2 + |\alpha_z|^2) \frac{N(N+2)}{2} + 2 \operatorname{Re}(\alpha_y^* \alpha_z Q). \quad (\text{B2})$$

The elements  $\Gamma_{C,yy}$  and  $\Gamma_{C,zz}$  can be obtained in a similar way, after relabeling the coordinates. Clearly, for  $(\alpha_x, \alpha_y, \alpha_z) = (1, 0, 0)$ , the state Eq. (32) corresponds to the  $D_x$  point in Fig. 3. Similarly,  $(\alpha_x, \alpha_y, \alpha_z) = (0, 1, 0)$  and  $(0, 0, 1)$  correspond to the  $D_y$  and  $D_z$  points, respectively. With an appropriate choice of phases for  $\alpha_i$ , a state with  $|\alpha_x| = |\alpha_y| = |\alpha_z|$  corresponds to the center of the  $D_x, D_y, D_z$  triangle. Moreover, a state with  $\alpha_x = i\alpha_y$  and  $\alpha_z = 0$  corresponds to a point halfway between  $D_x$  and  $D_y$ . In a similar manner, states of the form Eq. (32) can be obtained for the points halfway between  $D_x$  and  $D_z$ , and  $D_y$  and  $D_z$ .

Similar arguments show that with the appropriate choice of the absolute values and phases of  $\alpha_k$ , it is possible to get all the matrices,

$$\begin{aligned} \Gamma_c = & \alpha'_x \operatorname{diag} \left( 0, \frac{N(N+2)}{2}, \frac{N(N+2)}{2} \right) \\ & + \alpha'_y \operatorname{diag} \left( \frac{N(N+2)}{2}, 0, \frac{N(N+2)}{2} \right) \\ & + \alpha'_z \operatorname{diag} \left( \frac{N(N+2)}{2}, \frac{N(N+2)}{2}, 0 \right), \end{aligned} \quad (\text{B3})$$

with  $0 \leq \alpha'_i \leq 1$  and  $\alpha'_x + \alpha'_y + \alpha'_z = 1$ . That is, we can get any point corresponding of the  $D_x, D_y, D_z$  triangle in Fig. 3.

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